

Statistical properties of one dimensional attractive Bose gas

Przemysław Bienias,^{1,2} Krzysztof Pawłowski,^{3,4} Mariusz Gajda,⁵ and Kazimierz Rzażewski^{3,6,4}

¹ *College of Inter-Faculty Individual Studies in Mathematics and Natural Sciences,
University of Warsaw, ul. Żwirki i Wigury 93, 02-089 Warszawa, Poland*

² *Faculty of Physics, Warsaw University, ulica Hoża 69, PL-00-681, Warsaw, Poland*

³ *Center for Theoretical Physics, Polish Academy of Sciences, Aleja Lotników 32/46, 02-668 Warsaw, Poland*

⁴ *5. Physikalisches Institut, Universität Stuttgart, Pfaffenwaldring 57, 70550 Stuttgart, Germany*

⁵ *Institute of Physics, Polish Academy of Sciences, Aleja Lotników 32/46, 02-668 Warsaw, Poland*

⁶ *Faculty of Mathematics and Sciences, Cardinal Stefan Wyszyński University, ulica Dewajtis 5, 01-815, Warsaw, Poland*

(Dated: January 15, 2013)

Using classical field approximation we present the first study of statistical properties of one dimensional Bose gas with attractive interaction. The canonical probability distribution is generated with the help of a Monte Carlo method. This way we obtain not only the depletion of the condensate with growing temperature but also its fluctuations. The most important is our discovery of a reduced coherence length, the phenomenon observed earlier only for the repulsive gas, known as quasicondensation.

PACS numbers: 67.85.Bc, 67.85.-d, 03.75.Hh, 05.30.Jp
Keywords: ultracold atoms, statistics

INTRODUCTION

Statistical properties of finite number of bosons confined by a trapping potential are intensively studied ever since the very first experiments on dilute gas Bose-Einstein condensates. Initially a full understanding has been reached of the statistics of the confined ideal Bose gas [1–6]. In this context criticism of the most widely used grand canonical ensemble has been raised. As it turns out it predicts unphysical fluctuations of the number of condensed atoms. Interactions, an essential aspect of Bose atoms physics make the problem of statistics rather complicated. In the simplest, although still academic case, that of the box with periodic boundary condition, at least we know a priori what is a condensate. It is the zero momentum component of the atomic field. In the box with periodic boundary conditions we know more: we know analytically the spectrum and the expressions for the collective Bogoliubov excitations. So, one can look at the statistics of the gas as that of noninteracting bosonic quasiparticles. This determines the statistical properties of the gas at least at low temperatures [7]. Extension of the method to higher temperatures is already significantly more complicated. The Bogoliubov-Popov spectrum depends on the actual number of the condensed atoms rather than on their total number. Thus, itself is a fluctuating variable. Several papers attempted to take this into account [8, 9]. Another serious problem is the assumption that quasiparticles do not interact. In fact they do and they have a finite (temperature dependent) lifetime. In a realistic harmonic trap we have no analytic formulae for the quasiparticles and also the condensate degree of freedom is known only a posteriori since it is a function of an interaction strength, a frequency of the harmonic trap, a number of atoms and a temperature of

the sample. In two recent papers [10, 11] we have shown how to overcome most of these problems. To this end we proposed to use the classical field description of the system generating the proper thermal equilibrium distribution using a classical Metropolis algorithm [12]. Details of classical field approaches are presented in [13, 14]. It is the purpose of this Letter to report the first study of statistical properties of an attractive Bose gas. Confined to a box it is unstable. Hence, this simple model makes no sense for the attractive gas. The situation is different in a harmonic trap. Limited attractive condensates do exist in two and three dimensions. The lithium-7 condensate was among the first ones to be observed [15]. It then has lead to creation of bright solitons [16, 17]. In strictly 1D systems the effective repulsion of the kinetic energy makes it stable for any number of particles. We therefore concentrate our attention on the one dimensional, finite, attractive Bose gas trapped in a harmonic potential. Thus the Hamiltonian of the system has the form:

$$H = \int \hat{\Psi}^\dagger(x) \left(\frac{p^2}{2m} + \frac{1}{2} m \omega_0^2 x^2 \right) \hat{\Psi}(x) dx + \frac{g}{2} \int \hat{\Psi}^\dagger(x) \hat{\Psi}^\dagger(x) \hat{\Psi}(x) \hat{\Psi}(x), \quad (1)$$

where $\hat{\Psi}$ is the atomic field operator, m is the mass of an atom, ω_0 is the frequency of the harmonic trapping potential and we are particularly interested in $g < 0$. The case of positive coupling constant was studied in detail in [11].

The classical fields approximation consists in replacement of the quantized atomic field by a c-number wave function. This wave function is then conveniently expanded as a sum over the harmonic oscillator wave functions and expansion coefficients α_n are classical stochas-

tic complex variables.

$$\Psi(x) = \sum_{n=0}^{n_{max}} \alpha_n \varphi_n(x) \quad (2)$$

where the oscillator eigenfunctions φ_n are chosen to correspond to a harmonic oscillator of the frequency ω . Note that ω is not necessarily equal to ω_0 .

Thus the energy functional:

$$E[\Psi] = \int \Psi^*(x) \left(\frac{p^2}{2m} + \frac{1}{2} m \omega_0^2 x^2 \right) \Psi(x) + \frac{g}{2} \int |\Psi(x)|^4, \quad (3)$$

takes a form:

$$E(\{\alpha_n\}) = \sum_{n=0}^{n_{max}} \hbar \omega n |\alpha_n|^2 + E_{int}(\{\alpha_i\}) + \sum_{n,n'=0}^{n_{max}} \frac{1}{2} m (\omega_0^2 - \omega^2) \langle n|x^2|n' \rangle \alpha_n^* \alpha_{n'} \quad (4)$$

with the interaction energy E_{int} being a quartic form in the variables α_n and $\langle n|x^2|n' \rangle$ is the matrix element in the oscillator basis.

We are going to look at the statistics of a sample composed of N atoms, so the amplitudes are subject to a constraint:

$$\sum_{n=0}^{n_{max}} |\alpha_n|^2 = N \quad (5)$$

Having converted a quantum statistical physics problem into classical one we then define a canonical equilibrium distribution of the amplitudes α_n as

$$P(\{\alpha_i\}) \propto \exp \left[-\frac{E(\{\alpha_i\})}{k_B T} \right] \quad (6)$$

An efficient algorithm generating this probability distribution has been invented by Metropolis [12].

It is easy to check with the help of Gross-Pitaevskii equation imaginary time propagation that the zero temperature attractive 1D Bose gas has a wave function which is very close to a Gaussian. Its width shrinks with growing number of condensed atoms and with growing absolute value of the negative coupling constant g . This is our guide to the optimal choice of the frequency ω defining the actual oscillator base in the expansion of the atomic classical field (2). Thus this frequency becomes a function of the coupling and of the temperature as they both determine the width of the condensate wave function. More precisely: as we have shown in [10] the classical fields reproduce faithfully the available analytically exact probability distribution of a harmonically confined ideal Bose gas for the cut-off parameter n_{max} satisfying:

$$n_{max} \hbar \omega_0 = k_B T \quad (7)$$

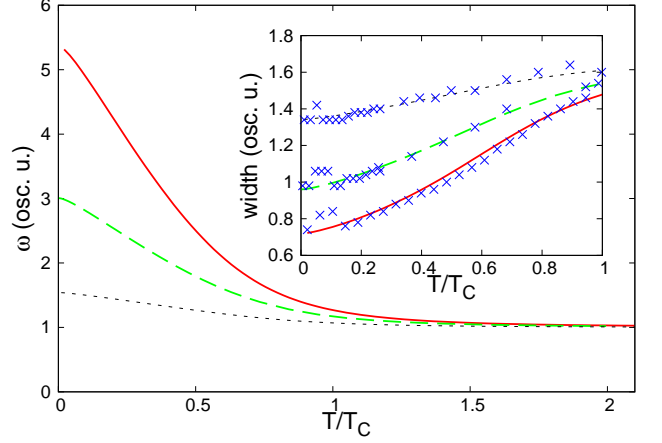


FIG. 1: (Color online) Effective frequency $\omega(N, T)$ for various values of the parameter g . The total number of atoms $N = 500$. Solid red line corresponds to $g = -0.01$, dashed green to $g = -0.0067$ and dotted black to $g = -0.003$. In inset we compare the FWHM of the Gaussian with effective ω with the width of the condensate (crosses).

where T is the temperature of the gas. A natural generalization of this condition for a weakly attractive 1D Bose gas is

$$n_{max} \hbar \omega(N, T) = k_B T \quad (8)$$

where the frequency $\omega(N, T)$ is chosen such that the corresponding Gaussian function describes the ground state of the Gross-Pitaevskii equation for N_0 interacting atoms. Strictly speaking N_0 should be chosen self consistently. In this Letter, however, we are satisfied with N_0 being a number of condensed **ideal** gas atoms at given temperature. The general rule for determination of the condensate wave function follows from the Onsager- Penrose [18] definition calling for the diagonalization of the one-particle density matrix

$$\rho_{i,j} = \langle \alpha_i^* \alpha_j \rangle = \sum_n \lambda_n \beta_i^*(n) \beta_j(n) \quad (9)$$

and identification of the condensate wave function with the eigenvector corresponding to the leading eigenvalue. As a cross-check we verified that our identification of base indeed yields the condensate wave function very close to the $n=0$ state.

It is clear that our choice of the frequency $\omega(N, T)$ makes it monotonically decrease with the temperature and tend to that of the empty trap: ω_0 . In Fig. 1 we illustrate this for several values of the coupling g . In the inset we also confirm a consistency of our choice of the base by comparing the width of the ground state of the Gross-Pitaevskii equation with the width of the actual condensate wave function computed via diagonalization of the one- particle density matrix. Small steps visible in the inset result from taking the solution of (8) for the cut-off as the nearest integer.

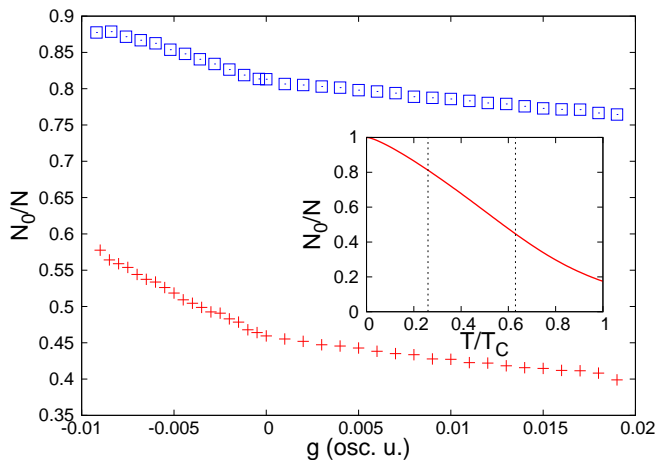


FIG. 2: (Color online) Number of atoms in the condensate as a function of the interaction strength g for two fixed temperatures $T = 0.26T_C$ (squares) and $T = 0.63T_C$ (crosses). These temperatures are indicated in the inset, where the decay of condensed atoms in the ideal gas is presented.

Multiparticle Hamiltonian with interactions dependent on the mutual distances between atoms with the harmonic potential trapping has the center of mass of the system obey interaction free Schrödinger equation decoupled from all other degrees of freedom. Thus this variable is entirely missing from any mean field approximation in which a nonlinear Gross-Pitaevskii equation describes the system. This does not have important consequences for the repulsive gas. In this case, the condensate is broader than the ground state of the harmonic potential, so the uncertainty over the position of the center is just a correction. The situation for attractive gas is different. Now, the condensate is spatially squeezed and the shot-to-shot variation of the position of the center of mass is at least as broad as the ground state of the empty harmonic potential [19, 20]. We therefore stress that our approach disregards this uncertainty, so the statistical properties derived here pertain to the reference frame of the center of mass of the system. Measurements in the center of mass system can meet some difficulties due to uncertainty of position of the center of mass. However, a single shot simultaneous detection of many particles resolves this problem [20].

In our classical field approximation effects of quantum fluctuations and quantum depletion are missing. The method is suitable for weakly interacting Bose gas in the quantum degenerate regime. Throughout this paper we use the oscillator units of position, energy and temperature, $\sqrt{\frac{\hbar}{m\omega_0}}$, $\hbar\omega_0$ and $\frac{\hbar\omega_0}{k_B}$ respectively. Hence a dimensionless coupling g is in units of $\sqrt{\frac{\hbar^3\omega_0}{m}}$.

All presented results are for $N = 500$ atoms and the temperature is in units of the characteristic temperature indicating transition to the quantum degenerate regime

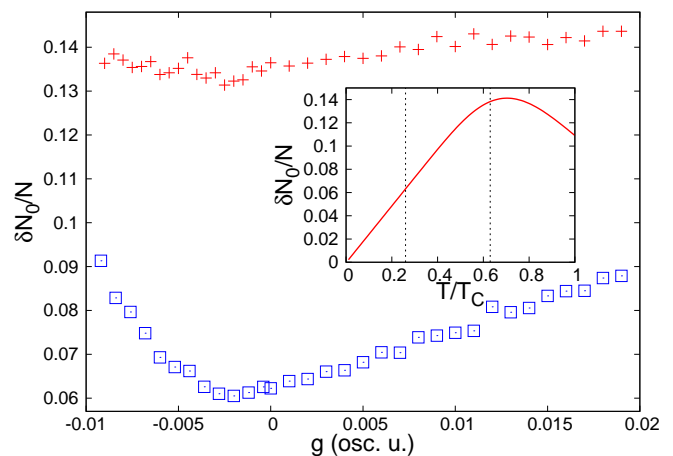


FIG. 3: (Color online) Dispersion of number of condensed atoms as a function of the interaction strength g for two fixed temperatures $T = 0.26T_C$ (squares) and $T = 0.63T_C$ (crosses). These temperatures are indicated in the inset, where the fluctuation of condensed atoms in the ideal gas is presented.

of a finite sample of an ideal Bose gas $N = T_C \ln(2T_C)$ where N is the total number of atoms [21]. In Fig. 2 we plotted a number of condensed atoms as a function of the coupling constant g for negative and also positive values (using the method explained in detail in [11]) for two temperatures, one very low ($T/T_C = 0.26$) and the other $T/T_C = 0.63$. Since the condensation depends on the local density of particles, the effects of positive and negative interactions are opposite. The first one swells the condensate reducing the local density with respect to the ideal gas while the other shrinks it enhancing the condensation process. Thus the number of condensed atoms is a monotonically decreasing function of the interaction parameter g as we go from the negative to the positive values. In Fig. 3 we present the relative fluctuations also as a function of changing sign g and for the same temperatures as in Fig. 2. General observation is that fluctuations of the condensed atoms number grow with the strength of the interaction in both negative and positive directions [27]. An unexpected feature, however is that the minimal fluctuations are observed not for $g = 0$ but for tiny attractive interaction of -0.0025 .

Perhaps the most interesting aspect of a 1D repulsive Bose gas predicted [22] and then observed experimentally [23, 24] is the phenomenon of a quasicondensation. It is a reduction of the coherence length to values smaller than the length of the condensate above a certain characteristic temperature. In the recent Letter [25] the correlation length has been related to the gray solitons formed during a rapid cooling process [26]. We have computed the temperature dependent correlation length for the attractive case and found that the quasicondensation occurs also in this case. This is probably the most important

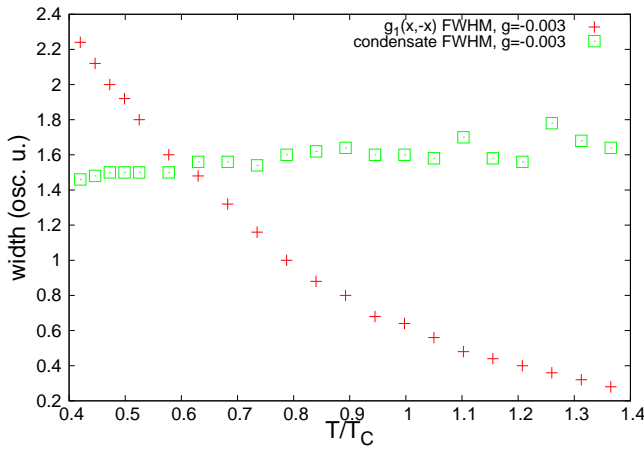


FIG. 4: (Color online) The width of the first order correlation function and the width of the condensate as functions of temperature.

result of this Letter. We note that one does not expect gray solitons to appear in rapidly cooled attractive Bose gas, so above mentioned connection to solitons certainly does not hold in this case. The first order correlation function is defined as

$$g_1(x, -x) = \frac{\langle \Psi(-x)^* \Psi(x) \rangle}{\langle |\Psi(x)|^2 \rangle} \quad (10)$$

In Fig. 4 we plot its full-width at half maximum as a function of temperature and compare it with the width of the condensate. The effect of quasicondensation is visible. It occurs at the temperature $T \approx 0.6T_C$.

Summarizing: We have presented the first study of equilibrium thermodynamics of the attractive 1D Bose gas trapped in a harmonic potential. Our results are for the canonical statistical ensemble, thus the temperature is a control parameter. The classical field approximation is used. The appropriate probability distribution is obtained numerically using a Monte Carlo technique. We found a reduced coherence length above some characteristic temperature. This phenomenon was previously known only for a repulsive gas.

This work was supported by Polish Government Funds for the years 2010-2012. Two of us (K.P. and K.Rz.) acknowledge financial support of the project "Decoherence in long range interacting quantum systems and devices" sponsored by the Baden-Württemberg Stiftung".

- [2] H. D. Politzer, Phys. Rev. A **54**, 5048 (1996).
- [3] M. Gajda and K. Rzażewski, Phys. Rev. Lett. **78**, 2686 (1997).
- [4] P. Navez, D. Bitouk, M. Gajda, Z. Idziaszek, and K. Rzażewski, Phys. Rev. Lett. **79**, 1789 (1997).
- [5] S. Grossmann and M. Holthaus, Opt. Express **1**, 262 (1997).
- [6] C. Weiss and M. Wilkens, Opt. Express **1**, 272 (1997).
- [7] S. Giorgini, L. P. Pitaevskii, and S. Stringari, Phys. Rev. A **54**, R4633 (1996).
- [8] A. A. Svidzinsky and M. O. Scully, Phys. Rev. Lett. **97**, 190402 (2006).
- [9] Z. Idziaszek and K. Rzażewski, Phys. Rev. A **68**, 035604 (2003).
- [10] E. Witkowska, M. Gajda, and K. Rzażewski, Phys. Rev. A **79**, 033631 (2009).
- [11] P. Bienias, K. Pawłowski, M. Gajda, and K. Rzażewski, Phys. Rev. A **83**, 033610 (2011).
- [12] N. Metropolis, A. W. Rosenbluth, M. N. Rosenbluth, A. H. Teller, and E. Teller, J. Chem. Phys. **21**, 1087 (1953).
- [13] M. Brewczyk, M. Gajda, and K. Rzażewski, Journal of Physics B: Atomic, Molecular and Optical Physics" **40**, R1 (2007).
- [14] N. P. Proukakis and B. Jackson, J. Phys. B: At. Mol. Opt. Phys. **41**, 203002 (2008).
- [15] C. C. Bradley, C. A. Sackett, J. J. Tollett, and R. G. Hulet, Phys. Rev. Lett. **75**, 1687 (1995).
- [16] K. E. Strecker, G. B. Partridge, A. G. Truscott, and R. G. Hulet, Nature **417**, 150 (2002).
- [17] L. Khaykovich, F. Schreck, G. Ferrari, T. Bourdel, J. Cubizolles, L. D. Carr, Y. Castin, and C. Salomon, Science **296**, 1290 (2002).
- [18] O. Penrose and L. Onsager, Phys. Rev. Lett. **104**, 576 (1956).
- [19] M. Gajda, M. A. Załuska Kotur, and J. Mostowski, Optics Express **8**, 106 (2001).
- [20] M. Gajda, Phys. Rev. A **73**, 023603 (2006).
- [21] W. Ketterle and N. J. van Druten, Phys. Rev. A **54**, 656 (1996).
- [22] D. S. Petrov, G. V. Shlyapnikov, and J. T. M. Walraven, Phys. Rev. Lett. **85**, 3745 (2000).
- [23] S. Dettmer, D. Hellweg, P. Ryytty, J. J. Arlt, W. Ertmer, K. Sengstock, D. S. Petrov, G. V. Shlyapnikov, H. Kreutzmann, L. Santos, et al., Phys. Rev. Lett. **87**, 160406 (2001).
- [24] J. Esteve, J.-B. Trebbia, T. Schumm, A. Aspect, C. I. Westbrook, and I. Bouchoule, Phys. Rev. Lett. **96**, 130403 (2006).
- [25] E. Witkowska, P. Deuar, M. Gajda, and K. Rzażewski, Phys. Rev. Lett. in press Phys. Rev. Lett.
- [26] W. H. Zurek, Phys. Rev. Lett. **102**, 105702 (2009).
- [27] For $T > 0.75$ the fluctuations start to decrease, see Fig. 3 in [11]